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Irregular conformal block and spectral curve: Matrix model approach

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Virasoro primary and descendents

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Highest weight state:

$$L_0|\Delta>=\Delta|\Delta>, \qquad L_k|\Delta>=0 \text{ when } k>0$$

Descendent state:

$$L_0\Big(L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Delta>\Big)$$
$$=(n_1+n_2+\cdots n_k+\Delta)\Big(L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Delta>\Big)$$

Irregular state

- Simultaneous eigenstate of positive Virasoro generators;

$$[L_m, L_n] = (m-n)L_{m+n} \text{ for } m, n \geq 0.$$

- Irregular state with rank n

$$L_k|I^{(n)}\rangle = 0 \quad \text{for} \quad 2n < k$$
$$L_k|I^{(n)}\rangle = \Lambda_k|I^{(n)}\rangle \quad \text{for} \quad n \leq k \leq 2n$$

- Irregular state is not the eigenstate of L0, but is a special combination of descendents (Gaiotto state or Whittaker state).

Rank 1 (L1 and L2 simultaneous eigenstate)

$$L_2|\Delta,\Lambda,\xi\rangle = -\Lambda^2|\Delta,\Lambda,\xi\rangle\,, \quad L_1|\Delta,\Lambda,\xi\rangle = -2\xi\Lambda|\Delta,\Lambda,\xi\rangle$$

$$|\Delta,\Lambda^2\rangle = \sum_{\ell}\Lambda^{\ell}\; w_n$$

$$w_0 = |\Delta\rangle$$

$$w_1 = -\frac{\xi}{\Delta}L_{-1}|\Delta\rangle$$

$$w_2 = \frac{(c\xi^2+\Delta(3+8\xi^2))L_{-1}^2 - 2\Delta(1+2\Delta+6\xi^2)L_{-2}}{4\Delta(2\Delta+c+16\Delta^2-10\Delta)}|\Delta\rangle$$

Gaiotto (2009)

More details in Marshak, Minorov, Morozov (2009)

Irregular state of rank n > 1

$$L_k|G_{2n}\rangle = \Lambda_k|G_{2n}\rangle \text{ for } n \le k \le 2n$$

$$\Lambda_n = \Lambda m, \quad \Lambda_k = \Lambda^{k/n} a_{2n-k} \text{ when } n < k \le 2n$$

$$|G_{2n}\rangle = \sum_{\ell=0}^{\infty} \sum_{\ell_p} \Lambda^{\ell/n} \prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} b_i^{\ell_i} m^{\ell_n}$$

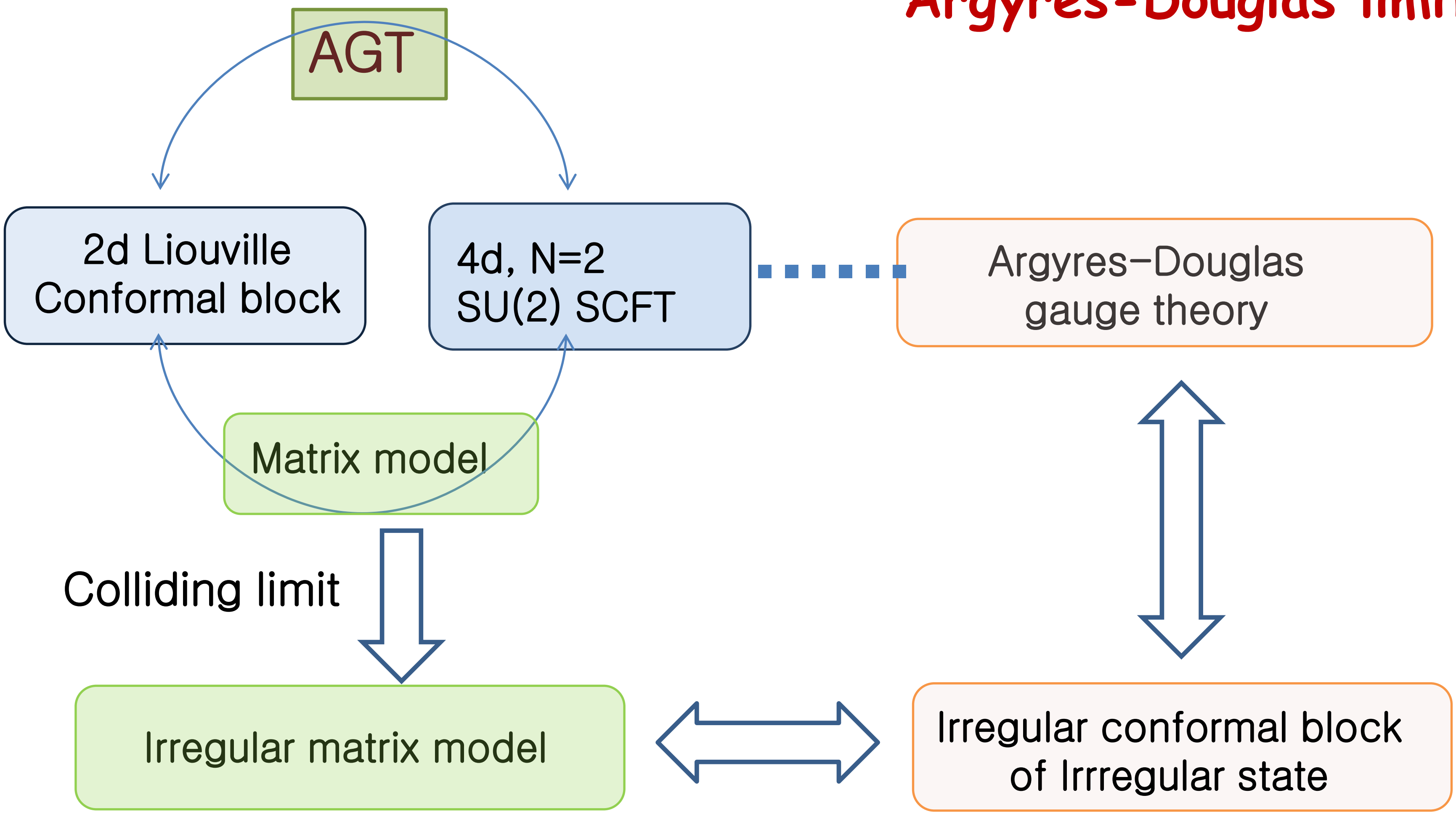
$$Q_\Delta^{-1}(2n^{\ell_{2n}}(2n-1)^{\ell_{2n-1}} \cdots 2^{\ell_2} 1^{\ell_1}; Y)\ L_{-Y}\,|\Delta\rangle\,,$$

$$Q_\Delta(Y;Y') = \langle\Delta|L_{Y'} L_{-Y}|\Delta\rangle$$

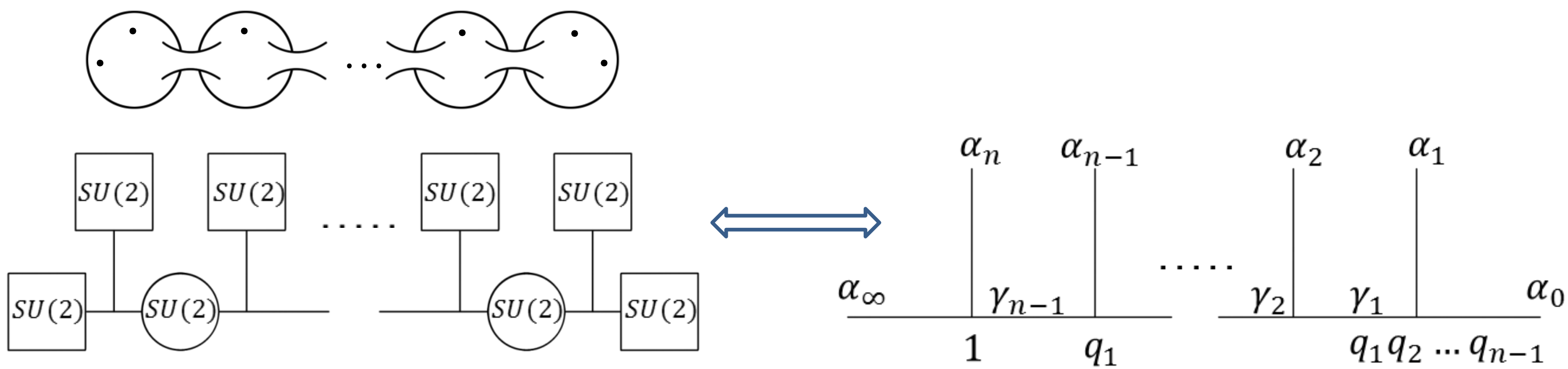
(Kanno, Maruyoshi, Shiba, Taki. 2013)

- The state contains more information than eigen-values!
- What is $b_i = ?$

Irregular state,
N=2 super Yang-Mills gauge theory,
Argyres-Douglas limit



AGT correspondence



Ω-deformation parameter $\epsilon_{1,2}$	Liouville parameter b
Marginal gauge couplings τ_k	Positions of vertex operators
Mass parameters m_a	External momenta α_a
Coulomb branch parameters a_i	Internal momenta γ_i

- 4d N=2 SU(2) SCFT vs 2d Liouville theory
- Instanton partition function vs Liouville conformal block

Plan of talk

1. Virasoro Irregular Conformal Block
2. W3 Irregular Conformal Block
3. Summary and discussion

With S. Choi and H. Zhang 1510.09060 (JHEP 03(2016)118

1411.4453 (PLB 742 (2015) 50

With H. Zhang [1504.07910(JHEP 1507 (2015) 163) ; 1506.03561 (JHEP 1509 (2015) 097)]

With S. Choi [1312.5535 (JHEP 04(2014) 106), 1506.02421(JPHY A)]

with T. Nishinaka 1207.4480 (JHEP 10 (2012) 138)

1. Virasoro irregular conformal block

[1] Random matrix model and colliding limit
[2] Spectral curve and conformal symmetry
[3] Partition function and irregular conformal block

Realization of irregular state

- Use coherent state representation of Heisenberg algebra

$$a_k|I^{(m)}\rangle = c_k|I^{(m)}\rangle \quad \text{for} \quad 1 \leq k \leq m$$

$$a_k|I^{(m)}\rangle = 0 \qquad\qquad \text{for} \quad k > m$$

- Virasoro generator realization for $c = 1 + 6Q^2, \quad Q = b + 1/b$

$$L_k|I^{(m)}\rangle = \Lambda_k|I^{(m)}\rangle \quad \text{for} \quad m \leq k \leq 2m$$

$$\Lambda_k = (k+1)Q\,c_k - \textstyle\sum_{0\leq\ell\leq k} c_k c_{k-\ell}$$

$$L_k|I\rangle = (\Lambda_k + v_k)|I\rangle \quad \text{for} \quad 0 \leq k \leq m-1$$

$$v_k = \textstyle\sum_{0\leq\ell\leq m} \ell\; c_{\ell+k} \frac{\partial}{\partial c_\ell}$$

Gaiotto &Teschner (2012)

Singularity structure

$$T(z)|I^{(m)}\rangle = \sum_k \frac{L_k}{z^{k+2}}|I^{(m)}\rangle = \left(\sum_{k=m}^{2m} \frac{\Lambda_k}{z^{k+2}} + \sum_{k<m} \frac{\Lambda_k + v_k}{z^{k+2}}\right)|I^{(m)}\rangle$$

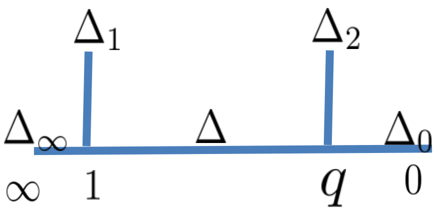
- Energy momentum tensor has poles of degree greater than 2 on irregular state (m=0 for the regular state).

- How to understand the singularity structure, including the differential operator v_k?

- We will try random matrix model approach originated from the Liouville conformal block!

Liouville conformal block and Selberg integral

- Liouville primary field: $V_\alpha(z) = e^{2\alpha\ \phi(z)}$; $\Delta_\alpha = \alpha(Q-\alpha)$
- Free field correlation: $\langle e^{2\alpha_1\phi}(z) e^{2\alpha_2\phi}(w)\rangle = (z-w)^{-2\alpha_1\alpha_2}$,

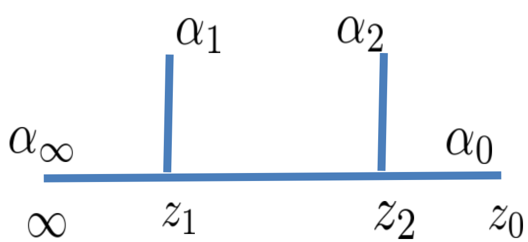


$$\propto\ q^{-2\alpha_2\alpha_0}(1-q)^{-2\alpha_1\alpha_2}$$

$$\times \int \left[\prod_{I=1}^{N} d\lambda_I\right] \prod_{I<J} (\lambda_I - \lambda_J)^{-2b^2} \prod_I (\lambda_I)^{-2b\alpha_0} (\lambda_I - q)^{-2b\alpha_2} (\lambda_I - 1)^{-2b\alpha_1}$$

- Screening operator: $\int d\lambda_I\ e^{2b\ \phi(\lambda_I)}$
- Neutrality condition: $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_\infty + bN = Q = b + 1/b$
- Back ground charge: $Q = b + \frac{1}{b}$

Random matrix model



$$\prod_{a<b}(z_a - z_b)^{-2\alpha_a \alpha_b} \times Z_4$$

Beta-deformed Penner-type matrix model:

$$Z_4 = \int \left[\prod_{I=1}^{N} d\lambda_I\right] \prod_{I<J} (\lambda_I - \lambda_J)^{2\beta} \; \exp\left(\frac{\sqrt{\beta}}{g}\sum_I V(\lambda_I)\right)$$

$$V(\lambda_I) = \sum_a \hbar\alpha_a \log(\lambda_I - z_a)$$

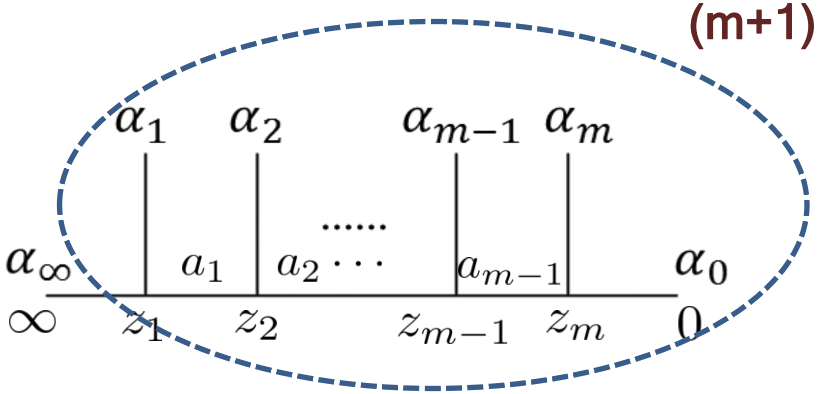
Dijkgraaf & Vafa (2009)
Itoyama & Oota (2010)

Equivalent parameter notations: $b = i\sqrt{\beta}, \quad g = i\hbar/2$

$$-\sqrt{\beta}/g = 2b/\hbar, \qquad \hbar b = 2g\sqrt{\beta}, \qquad 2g(\sqrt{\beta} - 1/\sqrt{\beta}) = \hbar Q$$

Colliding limit and irregular conformal block

- Many of vertex operators are put at the same point.
- The Liouville charges can be very large so that even after fusion, non-vanishing extra modes can exist.



(m+1) number of primary states are colliding:

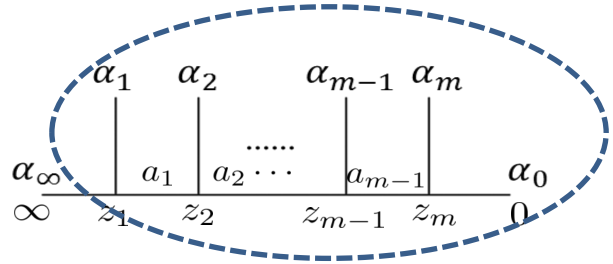
$$z_1, z_2, \cdots z_m \to 0$$

$$c_k = \sum_{a=1}^{m} \hbar \alpha_a z_a^k$$

Eguchi & Maruyoshi (2010);
Gaiotto & Teschner (2012)

Irregular matrix model of $< \Delta | I^{(m)} >$

- (m+1) number of primary states are colliding:



$$c_k = \sum_{a=1}^{m} \hbar \alpha_a z_a^k$$

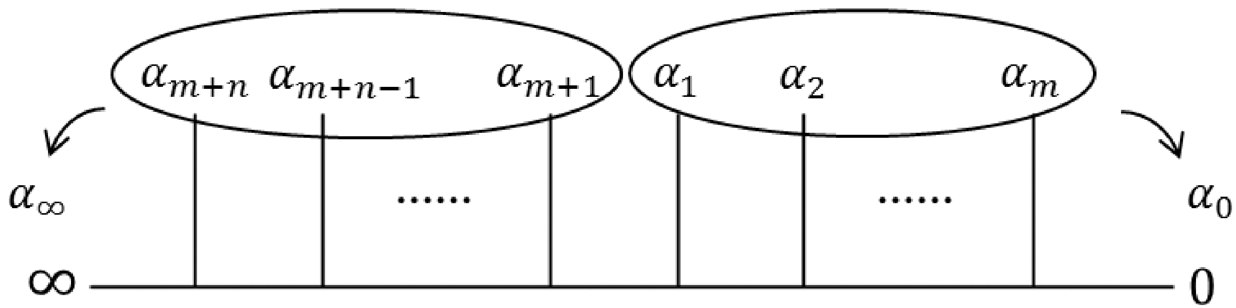
$$c_0 + \hbar b N + c_\infty = \hbar Q;$$

- Define the irregular matrix model with rank m

$$Z_{(0;m)} \equiv \left[\prod_{I=1}^{N} \int d\lambda_I \right] \Delta^{2\beta} \exp\left(\frac{\sqrt{\beta}}{g} \sum_I V(\lambda_I) \right)$$

$$\boxed{V(\lambda) = c_0 \log \lambda - \sum_{k=1}^{m} \frac{c_k}{k \lambda^k}}$$

Irregular matrix model of $\propto \langle I^{(n)} | I^{(m)} \rangle$



$c_{-\ell} = -\sum_{a=m+1}^{m+n} \hbar \alpha_a z_a^{-\ell}$

$c_k = \sum_{a=1}^{m} \hbar \alpha_a z_a^k$

$$V_{(n,m)}(\lambda) = c_0 \log z - \sum_{k=-n}^{-1} \frac{c_k}{k\lambda^k} + \sum_{k=1}^{m} \frac{c_k}{k\lambda^k}$$

How to evaluate the partition function

- Use the loop equation
- (Ward Identity of the conformal symmetry)

$$\frac{f(z)}{4} = W(z)^2 + V'(z)W(z) + g\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)W'(z) + g^2 W(z,z)$$

$$W(z) \equiv g\sqrt{\beta}\left\langle \sum_{I_1} \frac{1}{z - \lambda_{I_1}} \right\rangle = 1 \text{ point resolvent}$$

$$W(z,z) \equiv \beta \left\langle \sum_{I_1} \frac{1}{z - \lambda_{I_1}} \sum_{I_2} \frac{1}{z - \lambda_{I_2}} \right\rangle_{\text{conn}} = 2 \text{ point resolvent}$$

$$f(z) \equiv -\frac{\hbar b}{2} \sum_{I=1}^{N} \left\langle \frac{V'(z) - V'(\lambda_I)}{z - \lambda_I} \right\rangle \qquad \langle O \rangle = \frac{\int [d\lambda_I]\; O\; \Delta^{2\beta} e^{\frac{2b}{h}\sum_I V(\lambda_I)}}{\int [d\lambda_I] \Delta^{2\beta} e^{\frac{2b}{h}\sum_I V(\lambda_I)}}$$

Spectral curve and conformal symmetry

- Spectral curve (classical/NS limit of loop equation);

$$\hbar \to 0,\ \epsilon = \hbar Q \text{ finite}$$

$$\Longrightarrow x^2 + \epsilon x' - \xi_2(z) = 0; \qquad x \equiv 2W(z) + V'(z))$$

- $\xi_2(z)$ represents the conformal symmetry

$$\xi_2 := -V'^2 + \epsilon V'' - f = \sum_{k=0}^{2m} \frac{\Lambda_k}{z^{2+k}} - \sum_{a=0}^{m-1} \frac{d_a}{z^{2+a}} = \frac{\langle \Delta | T(z) | I^{(n)} \rangle}{\langle \Delta | I^{(n)} \rangle}$$

  - eigenvalue = $\Lambda_k = (k+1)\epsilon\, c_k - \sum_{r+s=k} c_r c_s$
  - expectation value = $d_a = v_a\Big(-\hbar^2 \log Z_{(n:m)} \Big)$

$$v_a = \sum_{0 \le \ell \le m} \ell\ c_{\ell+a} \frac{\partial}{\partial c_\ell}$$

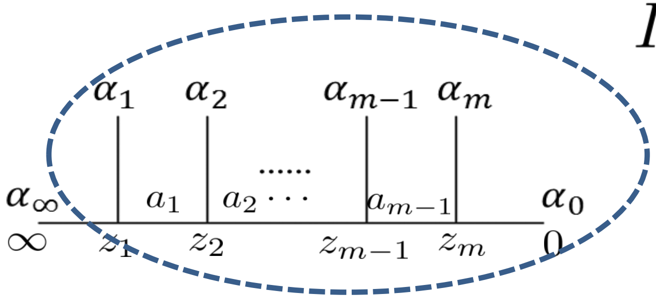
$$f(z) \equiv \sum_{a=0}^{m-1} \frac{d_a}{z^{2+a}} = -\frac{\epsilon}{2} \sum_{I=1}^{N} \left\langle \frac{V'(z) - V'(\lambda_I)}{z - \lambda_I} \right\rangle$$

Connection with gauge theory

- Seiberg-Witten differential $\lambda = x dz$ provides the scaling dimension so that $[x] + [z] = 1$.
- Use the Seiberg-Witten curve to find the scaling dimension.

$$[x] = 1 + \frac{1}{m}, \quad [z] = -\frac{1}{m}; \qquad [d_a] = 2 - \frac{a}{m}, \; [d_a] + [\lambda_{2m-a}] = 2$$

- Colliding limit produces more than a primary state; irregular conformal state which is generated by irregular operator:



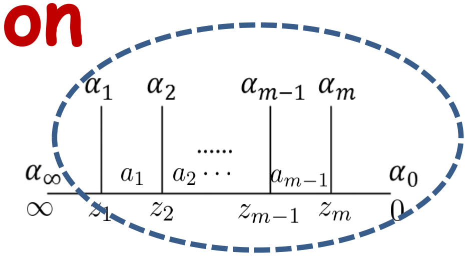
$$I_\alpha(z) = e^{2\alpha\ \Phi}(z);$$

$$\Phi(z) = \sum_{k=0}^{n} \frac{c_k}{k!} \frac{\partial^k \phi(z)}{\partial z^k}$$

Choi, Rim and Zhang(2015)
Polyakov and Rim (2016)

Flow equation and partition function

$$Z_{(0:m)}$$



- The partition function is the solution of flow equations

$$v_k\Big(-\hbar^2 \log Z_{(0:m)}\Big) = d_k; \qquad v_k := \sum_\ell \ell\, c_{\ell+k} \frac{\partial}{\partial c_\ell}$$

- The moment d_k of f(z) is found from the analytic structure of the spectral curve; $x(z)^2 + \epsilon x'(z) - \xi_2(z) = 0;$

$$\xi_2(z) = \sum_{k=0}^{2m} \frac{\Lambda_k}{z^{2+k}} - \sum_{k=0}^{m-1} \frac{d_k}{z^{2+k}}$$

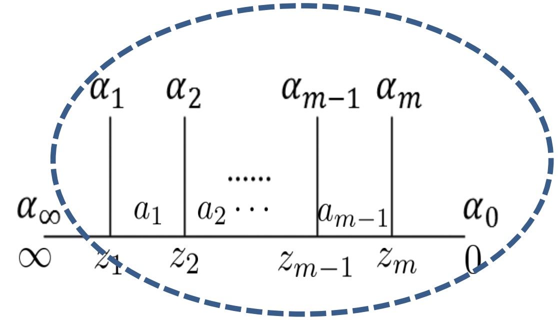
- Comments:

  (1) d_a satisfies the consistent conditions

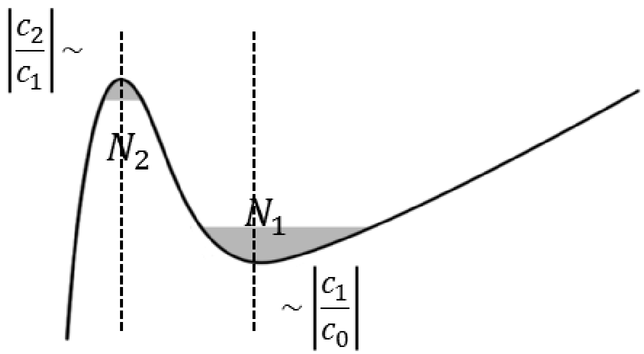
  (v_a satisfies the Virasoro commutation relation.)

  (2) d_a is the function of $\{c_\ell\}$

Some explicit form



- Rank 2 with 2-cut solution: $N = N_1 + N_2$

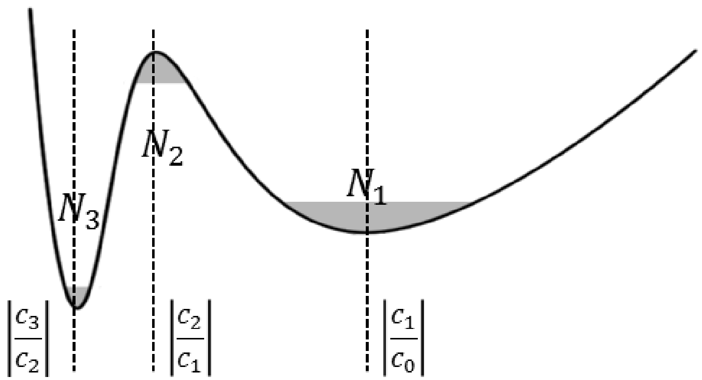


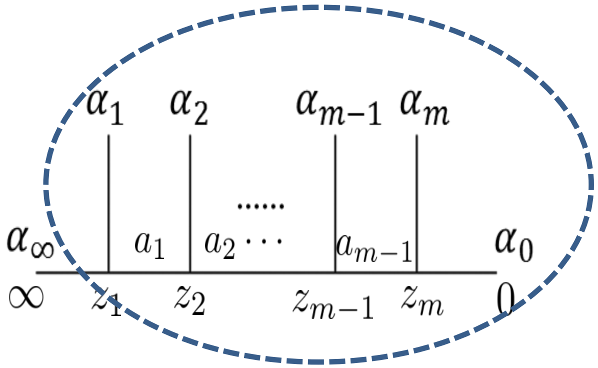
$$Z_N^{(2)} = (c_1)^{-h_1/\hbar^2} (\eta_1)^{-\frac{bN_2}{2}(3bN_2 + 4c_0/\hbar)} e^{-\frac{bN_2 c_0/\hbar}{\eta_1}} + \mathcal{O}(\eta_1)$$

$$h_1 = \hbar b N(\hbar b N + 2c_0), \qquad \eta_1 = c_0 c_2 / c_1^2$$

Nishinaka and Rim(2012)
Choi and Rim(2013)

- Rank 3 with 3-cut solution: $N = N_1 + N_2 + N_3$





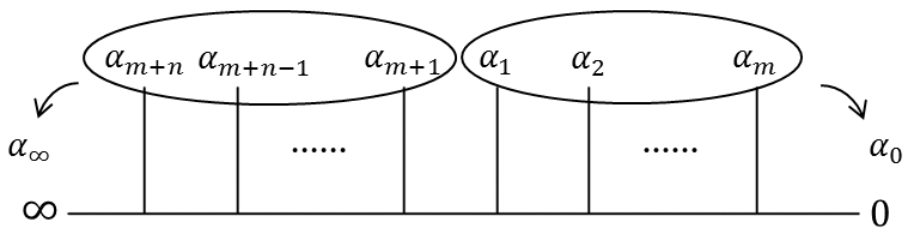
$$Z_N^{(3)} = (c_1)^{-h_1/\hbar^2} (\eta_1)^{-\frac{b}{2}(N_2(3bN_2+4c_0/\hbar)+N_3(3bN_3+4c_0/\hbar)+4bN_2N_3)}$$

$$(\eta_2)^{-2bN_3(bN_3+c_0/\hbar)} \; e^{-\frac{bc_0}{\eta_1\hbar}\left((N_2-N_3)+\frac{2N_3}{\eta_2}-\frac{N_3}{3\eta_2^2}\right)+O(\eta_1,\eta_2)} .$$

$$h_1 = \hbar bN(\hbar bN + 2c_0), \quad \eta_1 = c_0c_2/c_1^2, \quad \eta_2 = c_1c_3/c_2^2$$

Choi and Rim(2013)

Two point irregular conformal block from



$$\propto \langle I^{(n)} | I^{(m)} \rangle$$

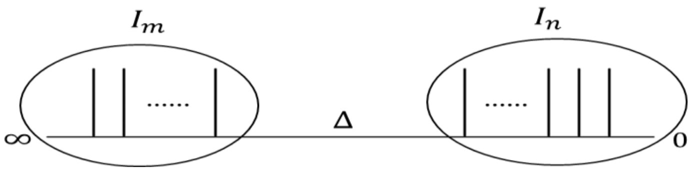
$$\frac{V_{(n,m)}(z)}{\hbar} = -c_0 \log z + \sum_{k=-n}^{-1} \frac{c_k}{k z^k} + \sum_{k=1}^{m} \frac{c_k}{k z^k}$$

- Irregular conformal block is proportional to the partition function of the irregular matrix model.

- One needs a proper normalization of inner product.

- Additional extra U(1) factor with careful treatment of the colliding limit of the inner product.

Step1: Potential as perturbations



$$\frac{V_{(n,m)}(z)}{\hbar} = -c_0 \log z + \sum_{k=-n}^{-1} \frac{c_k}{k z^k} + \sum_{k=1}^{m} \frac{c_k}{k z^k}$$

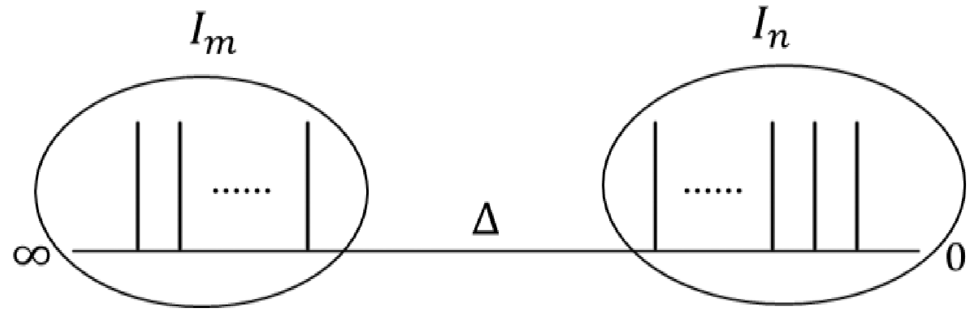
- Perturbation potential at z=0

$$\frac{1}{\hbar} V_0 = \sum_{I=1}^{N_0} \Big(- c_0 \log \lambda_I + \sum_{k=1}^{n} \frac{c_k}{k} \lambda_I^{-k} \Big) \, ; \quad \frac{1}{\hbar} \Delta V_0 = \sum_{I=1}^{N_0} \Big(\sum_{\ell=1}^{n} \frac{c_{-\ell}}{\ell} \lambda_I^{\ell} \Big) \, .$$

- Perturbation at z= infinity: change of variable $\lambda_J \to 1/\mu_J$

$$\frac{1}{\hbar} V_\infty = \sum_{J=1}^{N_\infty} \Big(- c_\infty \log \mu_J + \sum_{\ell=1}^{m} \frac{c_{-\ell}}{\ell} \mu_J^{-\ell} \Big) \, ; \quad \frac{1}{\hbar} \Delta V_\infty = \sum_{J=1}^{N_\infty} \Big(\sum_{k=1}^{n} \frac{c_k}{k} \mu_J^{k} \Big) \, .$$

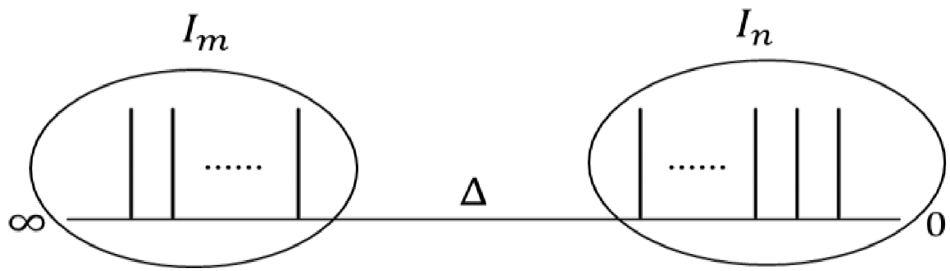
Step2: Normalization



Note that $\langle \Delta | I^{(m))} \rangle = 1 \Longleftrightarrow Z_{(0;m)} \neq 1$

$$\langle I^{(n)} | I^{(m)} \rangle \sim \frac{Z_{(n:m)}(c_0; \{c_k\}, \{c_{-\ell}\})}{Z_{(0:m)}(c_0; \{c_k\}) Z_{(0:n)}(c_\infty; \{c_{-\ell}\})}$$

Step3: extra U(1) factor at the colliding limit

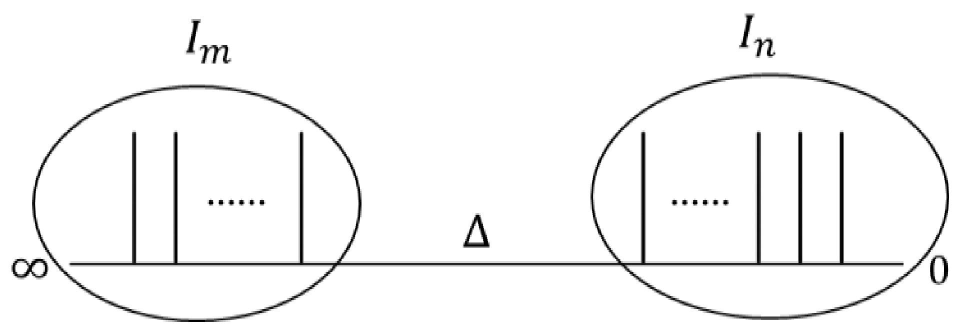


$$\propto \underbrace{\prod_{a=0}^{m+n+1} (z_a - z_b)^{-2\alpha_a \alpha_b}}_{e^{\zeta_{(n,m)}}} \times Z_{(n:m)}$$

$$e^{\zeta_{(n,m)}} \; ; \quad \zeta_{(m:n)} = \sum_k^{\min(m,n)} 2c_k c_{-k}/k$$

(2-point) Irregular conformal block

$$\langle I^{(n)}|I^{(m)}\rangle = \frac{e^{\zeta_{(m:n)}} Z_{(m:n)}(c_0; \{c_k\}, \{c_{-\ell}\})}{Z_{(0:n)}(c_0; \{c_k\}) Z_{(0:m)}(c_\infty; \{c_{-\ell}\})}$$
$$= \mathcal{F}_\Delta^{(m:n)}(\{c_{-\ell}\} : \{c_k\})$$



Comparison with algebraic construction

$$\mathcal{F}_{\Delta}^{(m:n)}(\{c_{-\ell}\}:\{c_k\}) = e^{\zeta_{(m:n)}} \left\langle \prod_{I,J} (1-\lambda_I \mu_J)^{2\beta} e^{-\frac{\sqrt{\beta}}{g}(\Delta V_0(\lambda_I)+\Delta V_\infty(\mu_J))} \right\rangle_{pert}$$

$$|G_{2n}\rangle = \sum_{\ell=0}^{\infty} \sum_{\ell_p} \Lambda^{\ell/n} \prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} b_i^{\ell_i} m^{\ell_n}$$
$$Q_{\Delta}^{-1}(2n^{\ell_{2n}}(2n-1)^{\ell_{2n-1}} \cdots 2^{\ell_2} 1^{\ell_1}; Y) L_{-Y} |\Delta\rangle \,,$$

$$\boxed{\Longrightarrow \Lambda^{k/n} b_i = \Lambda_i + v_i(\ln Z_{(0,n)})}$$

CRZ (1411.4453 (PLB 742 (2015) 50))

Coment on NS limit/classical limit

- Introduce a monic polynomial of degree N,

$$P(z) = \left\langle \prod_{I=1}^{N} (z - \lambda_I) \right\rangle;$$

- NS limit drops the multi-point resolvents; $W(z) = \frac{\epsilon}{2} \frac{d \log P}{dz};$

$$x(z)^2 + \epsilon x'(z) - \xi_2(z) = 0; \quad \epsilon = \hbar Q \quad x \equiv 2W(z) + V'(z))$$

$$\Longrightarrow \epsilon^2 P'' + 2\epsilon V'(z) P' - f(z) P = 0;$$

- d_k is determined algebraically

Rim and Zhang 1504.07910, 1506.03561)

2. W-irregular conformal block

[1] W symmetry and W irregular state
[2] spectral curve for W3 symmetry and partition function

W3 algebra and irregular state

W3 algebra (Fateev & Zamolodchikov,1987)

$$[L_p, L_q] = (p-q)L_{p+q} + \frac{c}{12}(p^3-p)\delta_{p,-q}$$

$$[L_p, W_q] = (2p-q)W_{p+q}$$

$$-\frac{2}{9}[W_p, W_q] = \frac{c}{3\cdot 5!}(p^2-1)(p^2-4)p\delta_{p,-q} + \frac{16}{22+5c}(p-q)\Lambda_{p+q}$$
$$+ (p-q)\left(\frac{1}{15}(p+q+2)(p+q+3) - \frac{1}{6}(p+2)(q+2)\right)L_{p+q}$$

$$\Lambda_p = \sum_{k=-\infty}^{\infty} :L_k L_{p-k}: + \tfrac{1}{5}x_p L_p; \quad x_{2\ell} = (\ell+1)(\ell-1), \; x_{2\ell+1} = (2+\ell)(1-\ell)$$

W3 irregular state
$$L_k|I^{(n)}\rangle = \Lambda_k|I^{(n)}\rangle \text{ for } n \le k \le 2n$$
$$W_k|I^{(n)}\rangle = \Lambda_k|I^{(n)}\rangle \text{ for } 2n \le k \le 3n$$

Kanno et al [1301.0721]
Choi & Rim [1506.02421]

Irregular conformal block with W3 symmetry

- (A2) Toda field theory and conformal block
- Colliding limit and irregular matrix model
- Loop equation and spectral curve contains (A2 case)
  - Quadratic: energy momentum tensor
  - Cubic: W_3 symmetry
- Partition function and irregular conformal block using the flow equations

C&R (1506.02421) CRZ(1510.09060)

Conformal block of Toda fields

- Vertex operators in (A2) Toda theory; $V_a(z_a) = e^{\vec{\alpha}_a \cdot \vec{\varphi}(z_a)}$

- Conformal block of n +m +2 primary operators is evaluated with two kinds of screening operators $e^{b\vec{e}_1 \cdot \vec{\varphi}(z_a)}$, $e^{b\vec{e}_2 \cdot \vec{\varphi}(z_a)}$

- Neutrality condition;

$$\vec{\alpha}_\infty + \sum_a \vec{\alpha}_a + bN\vec{e}_1 + bM\vec{e}_2 = 2Q(\vec{e}_1 + \vec{e}_2)$$

- Use the colliding limit to find the irregular matrix model

Irregular matrix model

$$\mathcal{Z} = \int \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} dx_i dy_j \Delta(x)^{2\beta} \Delta(y)^{2\beta} \Delta(x,y)^{-\beta} e^{\frac{2b}{\hbar}\left[\sum_{i=1}^{N} V_1(x_i) + \sum_{j=1}^{M} V_2(y_j)\right]},$$

$$V_1(z) = b_0 \log z - \sum_{k=-m,}^{-1} \frac{b_k}{k z^k} + \sum_{k=1}^{n} \frac{b_k}{k z^k}$$

$$V_2(z) = a_0 \log z - \sum_{k=-m}^{-1} \frac{a_k}{k z^k} + \sum_{k=1}^{n} \frac{a_k}{k z^k}$$

Neutrality condition; $b_0 + b_\infty + \epsilon(N - M/2) = \hbar Q$

$a_0 + a_\infty + \epsilon(M - N/2) = \hbar Q$

Spectral curve (classical/NS limit)

$$\epsilon = \hbar Q, \qquad \hbar/b \to 0$$

Drop off the two & three-point resolvents;

$$X_1^2 + X_2^2 - X_1 X_2 + \epsilon(X_1' + X_2') = -\xi_2$$

$$X_1^3 + \xi_2 X_1 + 3\epsilon X_1 X_1' + \epsilon^2 X_1'' = +\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'$$
$$X_2^3 + \xi_2 X_2 + 3\epsilon X_2 X_2' + \epsilon^2 X_2'' = -\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'$$

$$\frac{X_1}{2} := R_1(z) + t_1(z)\,, \qquad t_1(z) = \frac{2V_1'(z) + V_2'(z)}{3}\,,$$
$$\frac{X_2}{2} := R_2(z) - t_2(z)\,, \qquad t_2(z) = -\frac{V_1'(z) + 2V_2'(z)}{3}$$

Quadratic: Virasoro symmetry

$$X_1^2 + X_2^2 - X_1 X_2 + \epsilon(X_1' + X_2') = -\xi_2$$

$$\xi_2 = -2\epsilon(V_1'' + V_2'') - \frac{4}{3}(V_1'^2 + V_2'^2 + V_1' v_2') + f_1 + f_2$$

$$= \frac{\langle\Delta|T(z)|I_n\rangle}{\langle\Delta|I_n\rangle} \qquad T(z) = \sum_k \frac{L_k}{z^{k+2}}$$

$$L_k|I_n\rangle := \mathcal{L}_k|I_n\rangle; \qquad \mathcal{L}_k = \begin{cases} 0\,, & k > 2n \\ A_k\,, & n \le k \le 2n \\ A_k + v_k\,, & 0 \le k \le n-1\,. \end{cases}$$

$$A_k = 2\epsilon(k+1)(a_k + b_k) - \tfrac{4}{3}\textstyle\sum_{k=r+s}(a_r a_s + b_r b_s + a_r b_s)$$

$$v_k = \textstyle\sum_{s>0} s\left(b_{s+k}\frac{\partial}{\partial b_s} + c_{s+k}\frac{\partial}{\partial c_s}\right)$$

Cubic: W_3 symmetry

$$X_1^3 + \xi_2 X_1 + 3\epsilon X_1 X_1' + \epsilon^2 X_1'' = +\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'$$

$$X_2^3 + \hbar^2 \xi_2 X_2 + 3\epsilon X_2 X_2' + \epsilon^2 X_2'' = -\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'$$

$$\xi_3 = \frac{\langle \Delta | W(z) | I_n \rangle}{\langle \Delta | I_n \rangle} = \sum_{k=-3n}^{3m} \frac{B_k}{z^{k+3}} - \sum_{k=-2n}^{2m} \frac{e_k}{z^{k+3}}$$

$W(z) = \sum_k \frac{W_k}{z^{k+3}}$ is the W_3 current.

B_k $(2n \le k \le 3n)$ is the W_k eigenvalue of the irregular state.

$$e_k = \mu_k \left(-\hbar^2 \ln Z_{(m;n)}\right)$$

Flow equation ($n \le k \le 2n-1$) with

$$\mu_k = \sum_{k=r+s-t;\ t>0} \sqrt{3}t \left((a_r a_s + 2a_r b_s)\frac{\partial}{\partial a_t} - (b_r b_s + 2a_r b_s)\frac{\partial}{\partial b_t} \right)$$

Spectral Curve (Є=0)

$$X_1^3 + \xi_2 X_1 = +\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\hbar^2\xi_2'$$
$$X_2^3 + \xi_2 X_2 = -\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\hbar^2\xi_2'$$

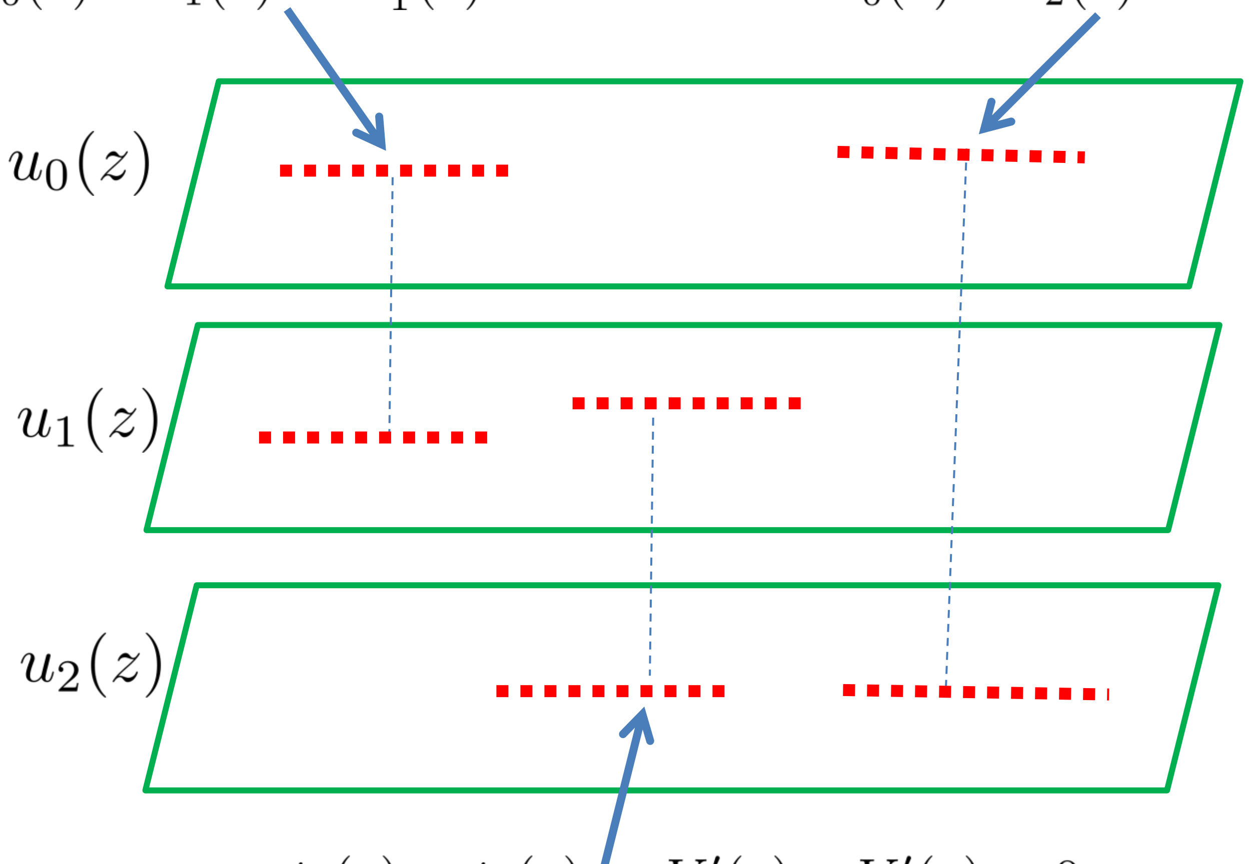
$$u_1(z) = \frac{X_1}{2} = R_1(z) + t_1(z)\,, \qquad t_1(z) = \frac{2V_1'(z) + V_2'(z)}{3}\,,$$
$$u_2(z) = -\frac{X_2}{2} = -R_2(z) + t_2(z)\,, \quad t_2(z) = -\frac{V_1'(z) + 2V_2'(z)}{3}$$
$$u_0(z) = -u_1(z) - u_2(z)\,, \quad t_0(z) = -t_1(z) - t_2(z)$$

$$\prod_{i=0}^{2}(u - u_i(z)) = u^3 + \frac{\hbar^2\xi_2(z)}{4}u - \frac{\hbar^3\xi_3(z)}{12\sqrt{3}} = 0\,.$$

Covering space and cut structure

$$t_0(z) = t_1(z) \Rightarrow V_1'(z) = 0.$$

$$t_0(z) = t_2(z) \Rightarrow V_2'(z) = 0$$



$$t_1(z) = t_2(z) \Rightarrow V_1'(z) + V_2'(z) = 0.$$

Simplest partition function (rank 1)

Flow equations:

$$v_0(-\hbar^2 \log Z_{0;1}) = d_0\,, \quad \mu_1(\hbar^2 \log Z_{0;1}) = e_1$$

Partition function:

$$\begin{aligned} Z_{(0:1)} = & a_1^{-(2\epsilon a_0 M_1+\epsilon^2 M_1(M_1-1))/\hbar^2} b_1^{-(2\epsilon b_0 N_1+\epsilon^2 a_0 N_1(N_1-1))/\hbar^2} \\ & \times (a_1+b_1)^{-(2\epsilon N_2(a_0+b_0)+\epsilon^2 N_2(N_2-3))/\hbar^2} \\ & \times (a_0 b_1 - a_1 b_0)^{-\epsilon^2(N_1 N_2+N_2+M_1 N_2-M_1 N_1)/\hbar^2} \end{aligned}$$

W3 irregular conformal block

$$\mathcal{F}_{\Delta}^{(m:n)}(\{a_{-k}, b_{-k} : a_k, b_k\}) = \frac{e^{\zeta_{(m:n)}} Z_{(m:n)}(a_0, b_0; \{a_\ell, b_\ell\})}{Z_{(0:n)}(a_0, b_0; \{a_k, b_k\}) Z_{(0:m)}(\bar{a}_0, \bar{b}_0; \{\bar{a}_k, \bar{b}_k\}))}.$$

$$\mathcal{F}_{\Delta}^{(1:1)} = 1 - \frac{1}{9\hbar^2 \left(4\omega_0^2 + \Delta^2(4\Delta - 3\epsilon^2)\right)} \Big[8\Delta\omega_{-1}\omega_1 + 12\omega_0(\omega_{-1}\ell_1 + \omega_1\ell_{-1}) - \frac{9}{2}\Delta\ell_{-1}\ell_1(4\Delta - 3\epsilon^2) \Big],$$

$$\Delta = -\tfrac{4}{3}(\alpha^2 + \alpha\beta + \beta^2) + 2\epsilon(\alpha + \beta)$$

$$\alpha = a_0 + \epsilon(M_0 - N_0/2),\ \beta = b_0 + \epsilon(N_0 - M_0/2)$$

$$\ell_1 = \frac{\langle \Delta | L_1 | I^{(1)} \rangle}{\langle \Delta | I^{(1)} \rangle} = -\tfrac{4}{3}(a_0(2a_1 + b_1) + b_0(a_1 + 2b_1)) + 4\epsilon(a_1 + b_1)$$

$$\omega_0 = \frac{\langle \Delta | W_0 | I^{(1)} \rangle}{\langle \Delta | I^{(1)} \rangle} - \tfrac{1}{3\sqrt{3}}(\alpha - \beta)(4\alpha + 2\beta - 3\epsilon)(2\alpha + 4\beta - 3\epsilon)$$

$$\omega_1 = \frac{\langle \Delta | W_1 | I^{(1)} \rangle}{\langle \Delta | I^{(1)} \rangle} = B_1 - e_1$$

Summary and discussion

- Irregular matrix model is used to find the loop equation (spectral curve) for the Virasoro (and W) irregular conformal block.
- Irregular conformal block is reconstructed using the flow equations in the spectral curve, which represents the conformal symmetry (or integrability in some literature) of the system.
- Multi-point Irregular conformal block on Riemannian manifold with/without boundary is not analyzed yet.
- More application will be useful, say, supersymmetrize irregular matrix model (for example, Polyakov and Rim [1604.08741]) or find the scaling behavior in the biological system.